

## Self duality and quantization

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The quantum theory of the free Maxwell field in Minkowski space is constructed using a representation in which the *self dual* connection is diagonal. Quantum states are now holomorphic functionals of self dual connections and a decomposition of fields into positive and negative frequency parts is unnecessary. The construction requires the introduction of new mathematical techniques involving “holomorphic distributions”. The method extends also to linear gravitons in Minkowski space. The fact that one can recover the entire Fock space – with particles of *both* helicities – from self dual connections alone provides independent support for a non-perturbative, canonical quantization program for full general relativity based on self dual variables.

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### 1. Introduction

Over the past three decades, Roger Penrose has provided us with several elegant mathematical techniques to unravel the structure of zero rest mass fields [1]. In particular, we have learnt from him that the description of these fields is especially simple if we decompose them using two-component spinors. For spin-1 and spin-2 fields – which mediate all four basic interactions – this amounts to focussing on the eigenstates of the Hodge-duality operator. It is striking indeed to see how, in the classical theory, the non-linear Yang–Mills and Einstein equations simplify in the self dual (or anti-self dual) sector. The richness of the mathematical structure of these solutions tempts one to conjecture that the notion of self duality should play a significant role in the quantization of these fields.

To see how this may come about concretely, let us first recall the relevant features of the theory of fields satisfying free relativistic equations in Minkowski space. The space of complex solutions to these equations provides us with unitary representations of the Poincaré group [2]. These are classified

by the values of the two Casimir operators, mass and spin, or, if the mass is zero, helicity. Let us focus on Maxwell fields. Then the construction yields four irreducible unitary representations consisting, respectively, of positive frequency self dual fields, positive frequency anti-self dual fields, negative frequency self dual fields and negative frequency anti-self dual fields. All of them have zero mass. The first and the fourth have helicity  $+1$  while the second and the third have helicity  $-1$  [3]. In the quantum theory, one generally chooses to work with the positive frequency polarization: one-photon states are represented by positive frequency fields and more general quantum states by entire holomorphic functionals on the one-photon Hilbert space. In this description, positive helicity photons correspond to self dual fields and the negative helicity ones to anti-self dual fields.

Note, however, that at least in principle one could have adopted another strategy: one could have restricted attention just to self dual fields.\* From the result [3] on the helicity of various sectors it would appear that, at least a priori, such a description should be viable. The positive frequency fields would now yield the helicity  $+1$  photons and the negative frequency ones the helicity  $-1$  photons. The strategy seems attractive because one would not have to *begin* by decomposing fields into positive and negative frequency parts, an operation which has no counterpart beyond the linear field theory in Minkowski space. Decomposition of fields into self dual and anti-self dual parts, on the other hand, is meaningful both for the non-linear gauge fields and general relativity. This then would be a concrete way in which self duality could play a key role in quantization.

Why has this avenue not been pursued in the literature? As we shall see in some detail, if one uses a self dual polarization in a straightforward way, one runs into a problem: although the polarization is well defined, unlike in the positive frequency case, it fails to be Kähler. More precisely, the situation is as follows. In the standard quantization procedure, one is naturally led to define the inner product on positive frequency fields  $F^+$  in terms of the symplectic structure  $\Omega: \langle F_1^+, F_2^+ \rangle := \Omega(F_1^+, F_2^+)$ . The same principles lead one, in the case of the self dual polarization, to use the above expression for the inner product replacing only the positive frequency fields by the self dual ones. However, this strategy now fails: the resulting norm is no longer positive definite. The origin of the problem is of course that in the self dual polarization, one works with both positive and negative frequency fields and the inner product given above yields negative norms on negative frequency fields. One may attempt to rescue the situation by changing the inner product, introducing a minus sign *by hand* on the negative frequency part of the self dual

\* Just as the description in terms of negative frequency fields simply mirrors the standard, positive frequency description, that in terms of anti-self dual fields would mirror the one in terms of self dual fields.

sector. This would of course resolve the problem of negative norms. However, now a new problem arises: the algebra of field operators is no longer faithfully represented on the resulting Hilbert space! Thus, if one wishes to work in a representation in which the self dual Maxwell connection is diagonal, one must modify the quantization procedure. In particular, a new input is needed to select the appropriate inner product.

The purpose of this paper is to supply the necessary modifications in the case of the free Maxwell field.

For several decades now there has been available a fully satisfactory quantum theory of photons in terms of positive frequency fields. It is therefore clear that, were we interested *only* in Maxwell fields, there is really no need for the alternate strategy mentioned above. The motivation for this work comes, rather, from the quarters of quantum general relativity. Through his non-linear graviton construction, Penrose [4] has taught us that unexpected simplifications arise when one deals with self dual gravitational fields in general relativity. Einstein's field equations suddenly become more transparent, readily malleable and fully manageable. While the chances of finding a general solution to the full Einstein equations still appear to be remote, for fifteen years we have known that, in spite of all the non-linearities, the self dual sector is completely integrable [4,5]! Therefore, it seems natural to base a non-perturbative approach to quantum general relativity on this sector [6]. Such an approach is indeed being pursued vigorously and has already led to some unexpected insights. (For recent reviews, see refs. [7,8].) To have a greater confidence in this approach, however, it is necessary to verify that the main physical ideas and mathematical techniques it uses are viable in familiar theories as well. It is in this spirit that we wish to recover here the standard Fock description of photons, working, however, in a representation in which the self dual Maxwell connection is diagonal. A similar construction is available also for linear gravitons [9]. However, in that case, a number of new ideas come into play. The case of the Maxwell field has the advantage that we can focus just on one issue: Can one carry out quantization in a self dual representation?

Section 2 is devoted to preliminaries. In the first part, section 2.1, we outline the quantization program we wish to follow. This is a simplified version of a more general program that is being used for quantum general relativity [7]; we have merely extracted the steps that are needed in the simpler, Maxwell case. In the second part, section 2.2, we first recall the canonical framework underlying Maxwell theory and then use the quantization program of section 2.1 to obtain the precise statement of what is meant by a self dual representation. Section 3 contains the main results. We find that the modes of the self dual Maxwell field naturally split into two parts, which can, at the end, be identified with the two helicity states. (It is important to note that an explicit decomposition

of the field into its positive and negative frequency parts is *not* carried out anywhere in the construction.) We show in section 3.1 that quantization of positive helicity states is straightforward within the framework of the program although the final description is unconventional in certain respects. In section 3.2, we show that the negative helicity states can also be quantized using the general framework of the program. However, now the steps involved are more subtle and require new mathematical tools. The final picture is then summarized in section 3.3. We conclude in section 4 by discussing some of the ramifications of these results.

In this paper we use a canonical approach based on space-like three-surfaces. An analogous treatment of the Maxwell field on null planes is given in Josh Goldberg's contribution to this volume.

## 2. Preliminaries

Our primary aim in this paper is to use Maxwell fields as a probe to test certain aspects of a non-perturbative approach [6–8] to quantum gravity based on self dual fields. Therefore, we will closely follow the quantization program developed there even though the steps involved may not appear to be the most natural ones from the strict standpoint of the Maxwell theory. In the first part of this section, we outline the general program emphasizing the points at which new input is needed for quantization. This program is based on the canonical quantization method and is applicable for a wide class of systems. In the second part, we focus on the Maxwell field and construct the structures needed in the program. We will then be able to give a detailed and precise formulation of what is meant by a self dual representation. The problem of constructing this representation will be taken up in the next section.

### 2.1. THE QUANTIZATION PROGRAM

Consider a classical system with phase space  $\Gamma$ . To quantize the system, we wish to follow an algebraic approach. We will proceed in the following steps. \*

(i) Choose a subspace  $S$  of the space of complex valued function(al)s on  $\Gamma$  which is closed under the Poisson bracket operation and which is large enough so that any well-behaved function(al) on  $\Gamma$  can be expressed as (possibly a limit of) a sum of products of elements of  $S$ . Elements of  $S$  are referred to as *elementary classical variables* and are to have unambiguous quantum analogs. For a non-relativistic particle moving in a potential, for example,  $\Gamma$  is just

\* For simplicity, we assume that there are no constraints. A more complete discussion of the program, including a treatment of constraints, is given in ref. [7].

$\mathbb{R}^6$  and the elementary classical variables are generally taken to be the three configuration variables  $q^i$  and their conjugate momenta  $p_i$ .

(ii) Associate with each  $f$  in  $\mathcal{S}$ , an abstract operator  $\hat{f}$ . Even though at this stage there is no Hilbert space for them to act upon, we will refer to the  $\hat{f}$  as *elementary quantum operators*. Consider the free associative algebra they generate and impose on it the (generalized) canonical commutation relations (CCR):

$$[\hat{f}, \hat{g}] = i\hbar\{\widehat{f, g}\}, \quad \forall f, g \in \mathcal{S}. \quad (2.1)$$

Denote the resulting associative algebra by  $\mathcal{A}$ . For the non-relativistic particle, (2.1) are just the standard Heisenberg commutation relations and elements of  $\mathcal{A}$  are simply sums of products of  $\hat{q}^i$  and  $\hat{p}_i$  with identifications implied by the CCR.

(iii) Introduce an involution operation,  $\star$ , on  $\mathcal{A}$  by first defining

$$(\hat{f})^\star = \widehat{f}, \quad \forall f \in \mathcal{S}, \quad (2.2)$$

where  $\widehat{f}$  is the complex conjugate of the elementary classical variable  $f$ , and extend the action of  $\star$  to all of  $\mathcal{A}$  by requiring that it satisfy the three defining properties of an involution: (i)  $(\hat{A} + \lambda\hat{B})^\star = \hat{A}^\star + \bar{\lambda}\hat{B}^\star$ ; (ii)  $(\hat{A}\hat{B})^\star = \hat{B}^\star\hat{A}^\star$ ; and (iii)  $(\hat{A}^\star)^\star = \hat{A}$ , where  $\hat{A}$  and  $\hat{B}$  are arbitrary elements of  $\mathcal{A}$  and  $\lambda$  is any complex number. Denote the resulting  $\star$ -algebra by  $\mathcal{A}^{(\star)}$ . Note that, at this stage,  $\mathcal{A}^{(\star)}$  is an abstract  $\star$ -algebra; the  $\star$ -operation does *not* correspond to Hermitian conjugation on any Hilbert space. For the non-relativistic particle,  $\mathcal{A}^{(\star)}$  is obtained simply by making each  $\hat{q}^i$  and each  $\hat{p}_i$  its own  $\star$ -adjoint.

(iv) Choose a linear representation of  $\mathcal{A}$  on a vector space  $V$ . The  $\star$ -relations are ignored in this step. One simply wishes to incorporate the linear relations between the operators and the CCR. For the non-relativistic particle, one may choose for  $V$  the space of smooth functions  $\Psi(\mathbf{q})$  with compact support on  $\mathbb{R}^3$ , represent  $\hat{q}^i$  by a multiplication operator and  $\hat{p}_i$  by  $i\hbar$  times a derivative operator.

(v) Introduce on  $V$  an Hermitian scalar product  $\langle \cdot, \cdot \rangle$  by demanding that the abstract  $\star$ -relations become concrete Hermitian-adjointness relations:

$$\langle \Psi, \hat{A}\Phi \rangle = \langle \hat{A}^\star\Psi, \Phi \rangle, \quad \forall \hat{A} \in \mathcal{A} \text{ and } \forall \Psi, \Phi \in V. \quad (2.3)$$

Note that (2.3) is now a condition on the choice of the inner product. The Hilbert space  $\mathcal{H}$  is obtained by taking the Cauchy completion of  $(V, \langle \cdot, \cdot \rangle)$ .

The program requires two external inputs: the choice of the space  $\mathcal{S}$  in step 1 and the choice of the representation in step 4. One may make “wrong” choices and find that the program cannot be completed (for examples, see ref.

[7] and also section 3.2 below). However, if the choices *are* viable – i.e., if the program can be completed at all – one is ensured of the uniqueness of the resulting quantum description in a certain well-defined sense [7]. For the non-relativistic particle, for example, the choices we made are viable and step 5 does indeed provide the standard  $L^2$  inner product on  $V$ . In the framework of this program, the text-book treatment of free fields in Minkowski space can be summarized as follows. The phase space  $\Gamma$  can be taken to be the space of (real) solutions to the field equations; smeared out fields provide elementary variables; the  $\star$ -relations say that each smeared out field operator is its own star; the representation space  $V$  is the space of holomorphic functionals of positive frequency classical fields; field operators are represented as sums of multiplication (creation) and derivative (annihilation) operators and the unique inner product which realizes the  $\star$ -relations is given by the Poincaré invariant Gaussian measure on the space of positive frequency fields (see, e.g., ref. [10]).

## 2.2. SELF DUAL VARIABLES FOR THE MAXWELL FIELD

Let us begin with a brief summary of the standard phase space formulation of Maxwell fields. Denote by  $\Sigma$  a space-like three-plane in Minkowski space. Thus,  $\Sigma$  is topologically  $\mathbb{R}^3$  and is equipped with a flat, positive definite metric  $q_{ab}$ . The configuration variable for the Maxwell field is generally taken to be the connection one-form  $A_a(\mathbf{x})$  – the vector potential for the magnetic field – on  $\Sigma$ . Its canonically conjugate momentum is the electric field  $E^a(\mathbf{x})$  on  $\Sigma$ . The fundamental Poisson bracket is:

$$\{A_a(\mathbf{x}), E^b(\mathbf{y})\} = \delta_a^b \delta^3(\mathbf{x}, \mathbf{y}). \quad (2.4)$$

The system has one first class constraint,  $\partial_a E^a(\mathbf{x}) = 0$ . One can therefore pass to the reduced phase space by fixing transverse gauge. For simplicity, let us choose this avenue. The true dynamical degrees of freedom are then contained in the pair  $(A_a^\top(\mathbf{x}), E_\top^a(\mathbf{x}))$  of transverse (i.e., divergence-free) vector fields on  $\Sigma$ . Denote by  $\Gamma$  the phase space spanned by these fields; now there are no constraints and we are working only with the true degrees of freedom. On  $\Gamma$ , the only non-vanishing fundamental Poisson bracket is:

$$\{A_a^\top(\mathbf{x}), E_\top^b(\mathbf{y})\} = \delta_a^b \delta^3(\mathbf{x}, \mathbf{y}) - \Delta^{-1} \partial_a \partial^b \delta^3(\mathbf{x}, \mathbf{y}), \quad (2.5)$$

where  $\Delta$  is the Laplacian operator compatible with the flat metric  $q_{ab}$ . It is convenient – although by no means essential – to work in the momentum

space. Then, the true degrees of freedom are contained in the new dynamical variables  $q_j(\mathbf{k})$ ,  $p_j(\mathbf{k})$  with  $j = 1, 2$  :

$$\begin{aligned} A_a^\top(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} (q_1(\mathbf{k}) m_a(\mathbf{k}) + q_2(\mathbf{k}) \bar{m}_a(\mathbf{k})), \\ E_\top^a(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} (p_1(\mathbf{k}) m^a(\mathbf{k}) + p_2(\mathbf{k}) \bar{m}^a(\mathbf{k})), \end{aligned} \quad (2.6)$$

where  $\sqrt{2}m_a = \partial_a\theta + i\sin\theta \partial_a\phi$  is a complex vector field in the momentum space which is transverse,  $m_a(\mathbf{k})k^a = 0$ , and normalized so that  $m_a(\mathbf{k})\bar{m}^a(\mathbf{k}) = 1$ . The Poisson brackets (2.5) are equivalent to:

$$\{q_i(-\mathbf{k}), p_j(\mathbf{k}')\} = -\delta_{ij} \delta^3(\mathbf{k}, \mathbf{k}'), \quad (2.7)$$

while the fact that  $A_a^\top(\mathbf{x})$  and  $E_\top^a(\mathbf{x})$  are real translates to the conditions:

$$\bar{q}_j(\mathbf{k}) = q_j(-\mathbf{k}), \quad \bar{p}_j(\mathbf{k}) = p_j(-\mathbf{k}). \quad (2.8)$$

We are interested in using a self dual representation. Let us therefore first construct the self dual connection from the pair  $(A_a^\top(\mathbf{x}), E_\top^a(\mathbf{x}))$ . If we denote by  $d_a^\top(\mathbf{x})$  the transverse vector potential of the *electric field*,

$$d_a^\top(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}} (p_1(\mathbf{k}) m_a(\mathbf{k}) - p_2(\mathbf{k}) \bar{m}_a(\mathbf{k})), \quad (2.9)$$

the self dual connection is given simply by:

$${}^+A_a^\top(\mathbf{x}) = -A_a^\top(\mathbf{x}) + id_a^\top(\mathbf{x}). \quad (2.10)$$

Following the procedure used in general relativity [6], we now want to use the pair  $({}^+A_a^\top(\mathbf{x}), E_\top^a(\mathbf{x}))$  as our basic variables. From the viewpoint of the Maxwell theory, this choice is rather strange. However, it is in terms of the analogous ‘‘hybrid’’ canonical variables – one of which is complex and the other real – that the non-perturbative quantization program for full, non-linear relativity is most easily formulated. Therefore, here we will work with this unconventional choice. Following (2.9), let us expand  ${}^+A_a^\top(\mathbf{x})$  in terms of its Fourier components. We have:

$${}^+A_a^\top(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}} (z_1(\mathbf{k}) m_a(\mathbf{k}) - z_2(\mathbf{k}) \bar{m}_a(\mathbf{k})), \quad (2.11)$$

with

$$z_1(\mathbf{k}) = -|\mathbf{k}|q_1(\mathbf{k}) + ip_1(\mathbf{k}), \quad z_2(\mathbf{k}) = |\mathbf{k}|q_2(\mathbf{k}) + ip_2(\mathbf{k}). \quad (2.12)$$

In terms of these dynamical variables, the basic Poisson brackets are given by:

$$\{q_i(-\mathbf{k}), z_j(\mathbf{k}')\} = -i\delta_{ij}\delta^3(\mathbf{k}, \mathbf{k}'), \quad (2.13)$$

and the “reality conditions” (2.8) become:

$$\bar{q}_j(\mathbf{k}) = q_j(\mathbf{k}),$$

$$\bar{z}_1(\mathbf{k}) = -z_1(-\mathbf{k}) - 2|\mathbf{k}|q_1(-\mathbf{k}), \quad \bar{z}_2(\mathbf{k}) = -z_2(-\mathbf{k}) + 2|\mathbf{k}|q_2(-\mathbf{k}). \quad (2.14)$$

To summarize, our basic dynamical variables will be  $(z_j(\mathbf{k}), q_j(\mathbf{k}))$ . They satisfy the Poisson bracket relations (2.13) and the reality conditions (2.14). The Hamiltonian for the Maxwell theory,  $H := \int_{\Sigma} d^3x (E^T \cdot E^T + B^T \cdot B^T)$  (where  $B^T$  is the magnetic field), can now be expressed as:

$$H = \int d^3\mathbf{k} \sum_j \bar{z}_j(\mathbf{k}) z_j(\mathbf{k}), \quad (2.15)$$

where  $\bar{z}_j(\mathbf{k})$  can be regarded as functionals of  $z_j(\mathbf{k})$  and  $q_j(\mathbf{k})$  given by (2.14).

With this machinery at hand, we can now give a precise formulation of the problem we want to analyze. In the quantization program, we wish to use  $z_j(\mathbf{k})$  and  $q_j(\mathbf{k})$  as the *elementary classical variables*. We can then carry out steps 2 and 3 of the program in a straightforward fashion and arrive at a  $\star$ -algebra  $\mathcal{A}^{(\star)}$ . We again need new input in step 4. We want to use a representation in which the self dual connection – and hence the operators  $\hat{z}_j(\mathbf{k})$  – are diagonal. The obvious choice is to use for the vector space  $V$  the space of polynomials  $\Psi(z_j(\mathbf{k}))$  (which are, in particular, entire holomorphic functionals) and represent the  $\hat{z}_j$  by multiplication operators. The representation of  $\hat{q}_j(\mathbf{k})$  is then dictated by the generalized CCR that result from (2.13). This is the self dual representation we are seeking. The key questions now are: Can step 5 be carried out to completion? Does there exist an inner product which implements conditions (2.3) which arise from the reality conditions (2.14)? Is the inner product unique? Are the resulting Hilbert spaces large enough to accommodate the two helicities of photons? And, finally, is the resulting quantum description equivalent to the standard Fock theory?



### 3. Quantum theory

The form of the Hamiltonian (2.14) suggests that the Maxwell field can be regarded as an assembly of harmonic oscillators, containing two oscillators (labelled by  $j$ ) per momentum vector  $\mathbf{k}$ . Note furthermore, that the oscillators with  $j = 1$  are completely decoupled from those with  $j = 2$ ; each set is separately closed under the Poisson bracket relations (2.13) and the reality conditions (2.14). Therefore, we can simplify our task by first examining each set separately and *then* combine the results we obtain.

Let us begin by reviewing [7] the situation with a single harmonic oscillator. The phase space  $\Gamma$  is now two-dimensional, co-ordinatized by the real functions  $q$  and  $p$ . Following eq. (2.7), let us choose the fundamental Poisson bracket to be  $\{q, p\} = -1$ . The analog of the complex, self dual variable is  $z = q + ip$  [see eq. (2.10)]. The idea now would be to regard  $(z, q)$  as the elementary classical variables. Together with constants, they are indeed closed under the Poisson bracket

$$\{q, z\} = -i, \quad (3.1)$$

as well as the reality conditions

$$\bar{q} = q, \quad \bar{z} = -z + 2q. \quad (3.2)$$

The Hamiltonian  $H := q^2 + p^2$  now becomes:

$$H = \bar{z}z \equiv (-z + 2q)z. \quad (3.3)$$

Let us compare this structure with the one we encountered in section 2.2. If we let  $q$  and  $z$  here be, respectively, the analogs of  $q_j(-\mathbf{k})$  and  $z_j(\mathbf{k})$  of section 2.2, we find that for  $j = 2$  the two sets are *completely* analogous. For  $j = 1$ , however, there is a discrepancy: while the Poisson brackets and the Hamiltonians match, the reality conditions differ by a sign in one of the terms. This difference will turn out to play a crucial role in the implementation of the quantization program.

We begin in section 3.1 with the simpler,  $j = 2$  case. We carry out the quantization program of section 2.1 step by step for the harmonic oscillator described by equations (3.1)–(3.3). In the second part, section 3.2, we examine the ramifications of the sign discrepancy in the reality condition for the  $j = 1$  modes. We will find that this sign difference forces one to enlarge the framework and allow as states *holomorphic distributions*. In the last part, section 3.3, we collect all these results and present a coherent quantum description of the Maxwell field in the self dual representation.

3.1. THE  $j = 2$  MODES

The first step in the quantization program of section 2.1 is the introduction of the space  $\mathcal{S}$  of elementary classical variables. For the harmonic oscillator considered above, the natural choice is the complex vector space spanned by the functions  $1, z, q$  on the real, two-dimensional phase space  $\Gamma$ : This space is closed under the Poisson bracket operations and clearly “large enough” since it provides a (complex) co-ordinatization of  $\Gamma$ . To generate the algebra  $\mathcal{A}$ , introduce, first, the elementary quantum operators,  $\hat{1}, \hat{z}, \hat{q}$ , and on the collection of their formal sums of formal products, impose the CCR:

$$[\hat{q}, \hat{z}] = i\hbar\{\widehat{q, z}\} \equiv \hbar. \quad (3.4)$$

Using eq. (3.2), the  $\star$ -relations are also straightforward to impose. Set:

$$\hat{q}^* = \hat{q}, \quad \hat{z}^* = -\hat{z} + 2\hat{q}. \quad (3.5)$$

It is easy to check that the resulting  $\star$ -algebra  $\mathcal{A}^{(*)}$  is isomorphic to the standard  $\star$ -algebra constructed from the operators  $\hat{1}, \hat{q}, \hat{p}$  that one finds in text-books.

The next step is to find a representation of this algebra. It is here that the presence of the hybrid variables  $(z, q)$  suggests a new avenue. Let  $V$  now be the vector space of entire holomorphic functions  $\Psi(z)$  and let the concrete operators representing  $\hat{q}$  and  $\hat{z}$  be

$$\hat{q} \cdot \Psi(z) = \hbar \frac{d\Psi(z)}{dz}, \quad \hat{z} \cdot \Psi(z) = z\Psi(z), \quad (3.6)$$

so that the canonical commutation relations (3.4) are satisfied. Now, the question is whether the last step, step 5, of the quantization program can be carried out successfully: Is there is an inner product on  $V$  which realizes the  $\star$ -relations (3.5)? Let us begin by introducing a general measure  $\mu(z, \bar{z})$  on the complex  $z$ -plane on which the wave functions are defined and set the inner product to be:

$$\langle \Psi(z) | \Phi(z) \rangle = i \int dz \wedge d\bar{z} \mu(z, \bar{z}) \overline{\Psi(\bar{z})} \Phi(z). \quad (3.7)$$

(The factor of  $i$  arises because  $dp \wedge dq \equiv \Omega = 2i dz \wedge d\bar{z}$ .) Positivity of norms requires that  $\mu(z, \bar{z})$  must be real. This condition ensures that the requirement that  $\hat{q}$  is its own Hermitian adjoint is satisfied if one chooses  $\mu$  of the form  $\mu \equiv \mu(z + \bar{z})$ . It only remains to impose the  $\star$ -relation on  $\hat{z}$  as a condition on the choice of the inner product. It turns out that this condition now determines

the form of  $\mu$  completely! Up to an overall multiplicative constant,  $\mu$  is given by:

$$\mu(z, \bar{z}) = \exp\left(-\frac{1}{4\hbar}(z + \bar{z})^2\right).$$

Thus, the Hilbert space of quantum states consists of entire holomorphic functions of  $z$  which are normalizable with respect to the inner product:

$$\langle \Psi(z) | \Phi(z) \rangle = i \int dz \wedge d\bar{z} \exp\left(-\frac{1}{4\hbar}(z + \bar{z})^2\right) \overline{\Psi(z)} \Phi(z). \quad (3.8)$$

Note that there is freedom to add to the expression of the operator  $\hat{q}$  any holomorphic function of  $z$ ; this addition will not alter the commutation relations. It is easy to work out the change in the measure caused by this addition and show that the resulting quantum theory is unitarily equivalent to the one obtained above.

The question now is whether the space of normalizable states is “sufficiently large”. The simplest way to analyze this issue is to relate our Hilbert space to the one used in the Bargmann quantization [11] of the harmonic oscillator. Given a  $\Psi(z)$  in our Hilbert space, set  $f(z) = \exp(-z^2/4\hbar) \Psi(z)$ . Then, the finiteness of the norm of  $\Psi(z)$  is equivalent to:

$$i \int dz \wedge d\bar{z} e^{-z\bar{z}/2\hbar} |f(z)|^2 < \infty. \quad (3.9)$$

Note that the left hand side is precisely the norm of  $f(z)$  in the Bargmann Hilbert space! [In particular, the integral converges for all polynomials  $f(z)$ .] Thus, there is a 1–1 correspondence between our quantum states  $\Psi(z)$  and the Bargmann states  $f(z)$ ; the space of normalizable states is indeed “large enough”. Let us translate the action of the operators defined in (3.6) to the space of Bargmann states  $f(z)$ , using the unitary transform  $\Psi(z) \mapsto f(z) = \exp(-z^2/4\hbar)\Psi(z)$ . We find:

$$\hat{q} \cdot f(z) = \hbar \frac{df(z)}{dz} + \frac{z}{2} f(z), \quad \hat{z} \cdot f(z) = z f(z). \quad (3.10)$$

Equations (3.10) are precisely the expressions of the operators  $\hat{q}$  and  $\hat{z}$  in the Bargmann representation. Thus, the representation we constructed using the hybrid  $(q, z)$ -variables in the quantization program is unitarily equivalent to the Bargmann representation. Finally, note that in both these representations  $\hat{z}$  is the creation operator and  $\hat{z}^*$  is the annihilation operator.

We conclude this subsection with a remark relating our  $z$ -representation to the standard Schrödinger representation of the quantum oscillator. Our choice of  $V$  and the representation (eq. 3.6) was motivated by the fact

that  $z$  is complex and  $\hat{q}$  and  $\hat{z}$  satisfy the canonical commutation relations. Note, however, that one can also arrive at this choice systematically [12] using the fact that the passage from  $(q, p)$  to  $(q, z)$  corresponds to a simple canonical transformation  $(q, p) \rightarrow (q, dF/dq + ip)$  with generating function  $F(q) = \frac{1}{2}q^2$ . Had we used the pair  $(q, p)$  as our basic variables, the algebraic quantization program would have led us, as indicated in section 2.1, to the Schrödinger representation of the harmonic oscillator. In this picture, the states are represented by square-integrable functions  $\Psi(q)$  of the real variable  $q$  and the basic operators are given by  $\hat{q} \cdot \Psi(q) = q\Psi(q)$  and  $\hat{p} \cdot \Psi(q) = i\hbar(d\Psi/dq)$ . The canonical transformation now lets us pass to the new “momentum” or  $z$ -representation via the usual transform between the configuration and the momentum representations:

$$\Psi(z) := \int dq \exp\left(\frac{zq}{\hbar} - \frac{q^2}{2\hbar}\right) \Psi(q). \quad (3.11)$$

The function  $\Psi(z)$  is clearly holomorphic and, given a square-integrable  $\Psi(q)$ , the integral converges for all (complex values of)  $z$ . Thus, the result of the transform of any Schrödinger wave function is an entire holomorphic function in the  $z$ -representation. Using the expressions of the operators  $\hat{q}$  and  $\hat{p}$  in the Schrödinger representation, and the definition  $\hat{z} = \hat{q} + i\hat{p}$  of  $\hat{z}$ , we can now transform the operators  $\hat{q}$  and  $\hat{z}$  from the  $q$  to the  $z$ -representation. The result is precisely eq. (3.6).

### 3.2. THE $j = 1$ MODES

In terms of  $(z_j(\mathbf{k}), q_j(\mathbf{k}))$ , the only difference in the  $j = 1$  and  $j = 2$  modes is in the reality conditions. Let us therefore consider again a single simple harmonic oscillator with hybrid phase space variables  $(z, q)$ , proceed as in section 3.1 to construct the quantum algebra  $\mathcal{A}$  using the CCR (3.4), but introduce the  $\star$ -relation via:

$$\hat{q}^* = \hat{q}, \quad \hat{z}^* = -\hat{z} - 2\hat{q}. \quad (3.12)$$

The only difference between (3.12) and the  $\star$ -relations (3.5) of section 3.1 is in the sign of the very last term. Using these new  $\star$ -relations, we can complete step 3 of the program and obtain a  $\star$ -algebra  $\overline{\mathcal{A}}^{(\star)}$ . Note that  $\mathcal{A}^{(\star)}$  and  $\overline{\mathcal{A}}^{(\star)}$  are constructed from the same associative algebra  $\mathcal{A}$ ; the difference is only in the involution operation  $\star$ . Since the  $\star$ -relations are ignored in the step 4 of the program, we can attempt to use the same strategy as in section 3.1. Let us then choose  $V$  to consist of entire holomorphic functions of  $z$  and represent the operators via (3.6). Then, the canonical commutation relations (3.4) are

satisfied and we have a representation of the quantum algebra  $\mathcal{A}$ . Our final task is to introduce on  $V$  an inner product so that the  $\star$ -relations (3.12) become concrete Hermitian-adjointness relations on the resulting Hilbert space. As before, let us first make the ansatz (3.7) and then attempt to determine the measure  $\mu(z, \bar{z})$  using (3.12). As before, the measure is uniquely determined. However, since there is a change in sign in the reality condition, the sign in the exponent of the measure is now the opposite of what it was in section 3.1. We obtain:

$$\mu(z, \bar{z}) = \exp\left(+\frac{1}{4\hbar}(z + \bar{z})^2\right). \quad (3.13)$$

Consequently, the arguments that led us in section 3.1 to the conclusion that the Hilbert space of normalizable states is infinite dimensional (and naturally isomorphic to the Bargmann Hilbert space), now implies that there are *no* (non-zero) entire holomorphic functions which are normalizable with respect to the inner product of (3.13)! Thus, the change in sign in the reality conditions make a crucial difference: a new strategy is now needed in the choice of the linear representation.\*

A solution to this problem is suggested by the following considerations. A simple calculation shows that the change in the sign in the reality conditions amounts to exchanging the creation and annihilation operators. Thus, while the operator  $\hat{z}$  served as the creator in section 3.1, if we can complete the quantization program, it would now serve as the annihilation operator. It must therefore map the vacuum to zero. This suggests that, if the program is to succeed, we need to represent the vacuum by a “holomorphic, delta distribution”  $\delta(z)$ . Excited states can then be built by acting on this vacuum repeatedly by  $\hat{q}$ .

Since the meaning of these holomorphic distributions is not a priori clear, let us make a brief detour to introduce some mathematical techniques that are needed. Let us begin by defining the holomorphic *generalized function* (or *distribution*)  $\delta(z)$ . It will be the complex linear mapping from the space of functions of the type  $\sum f_i(z)g_i(\bar{z})$ , where  $f_i(z)$  are entire holomorphic functions and  $g_i(\bar{z})$  are entire anti-holomorphic functions, to the space of

\* The new strategy is motivated by the transform from the  $q$  to the  $z$ -representation discussed at the end of section 3.1. Note also that the choice of the vector space  $V$  we are about to introduce would be necessary also in the Bargmann quantization, had the symplectic structure been of opposite sign, or, alternatively, if the symplectic structure had been the same but we had used anti-holomorphic wave functions. In either case, we would have found that the measure needed to ensure the correct reality conditions is  $\exp(+z\bar{z}/2\hbar)$ , whence no entire (anti-)holomorphic function would have been normalizable. We would then have to use for states the holomorphic distributions introduced below.

entire anti-holomorphic functions, given by:

$$\delta(z) \circ \sum_i f_i(z) g_i(\bar{z}) = \sum_i f_i(0) g_i(\bar{z}). \quad (3.14)$$

Next, we can define the anti-holomorphic distribution  $\delta(\bar{z})$  simply by taking the complex conjugate of  $\delta(z)$ . This new distribution has the action:

$$\delta(\bar{z}) \circ \sum_i f_i(z) g_i(\bar{z}) = \sum_i f_i(z) g_i(0). \quad (3.15)$$

From these two basic distributions, we can construct others. The product of a polynomial  $a(z, \bar{z})$  with a distribution  $\mathcal{F}(z, \bar{z})$  will be a new distribution, given by:

$$[a(z, \bar{z})\mathcal{F}(z, \bar{z})] \circ \sum_i f_i(z) g_i(\bar{z}) := \mathcal{F}(z, \bar{z}) \circ a(z, \bar{z}) \sum_i f_i(z) g_i(\bar{z}). \quad (3.16)$$

Finally, using the Leibnitz rule as a motivation, we define the derivative of a distribution  $\mathcal{F}(z)$ , as

$$\begin{aligned} \left[ \frac{d}{dz} \mathcal{F}(z) \right] \circ \sum_i f_i(z) g_i(\bar{z}) &:= \frac{d}{dz} \left( \mathcal{F}(z) \circ \sum_i f_i(z) g_i(\bar{z}) \right) \\ &\quad - \mathcal{F}(z) \circ \frac{d}{dz} \sum_i f_i(z) g_i(\bar{z}). \end{aligned} \quad (3.17)$$

The derivative with respect to  $\bar{z}$  is defined similarly. As an example, let us compute the derivatives of  $\delta(z)$ . We have:

$$\frac{d}{d\bar{z}} \delta(z) = 0,$$

$$\left[ \frac{d}{dz} \delta(z) \right] \circ \sum_i f_i(z) g_i(\bar{z}) = - \sum_i \frac{df_i(z)}{dz} \Big|_{z=0} g_i(\bar{z}). \quad (3.18)$$

Thus,  $\delta(z)$  is “holomorphic” and its derivative with respect to  $z$  is a distribution with the expected property. Finally, we notice that the product of the two distributions (3.14) and (3.15) is well defined; it is just the two dimensional  $\delta$ -distribution and therefore admits the standard integral representation:

$$\begin{aligned} [\delta(z)\delta(\bar{z})] \circ \sum_i f_i(z) g_i(\bar{z}) &= \sum_i f_i(0) g_i(0) \\ &\equiv \int dp \wedge dq \delta^2(q, p; 0, 0) \sum_i f_i(z) g_i(\bar{z}), \end{aligned} \quad (3.19)$$

where we have used  $z = q + ip$ . Thus, one can regard  $\delta(z)$  as the “holomorphic square root” of the standard two-dimensional  $\delta$ -distribution on the two-plane, picked out by the complex structure.

With this machinery at hand, let us proceed with the quantization program. In step 4 of the program let us use, as the carrier space for the representation, the space  $V$  spanned by the holomorphic distributions of the type  $\Psi(z) = \sum a_n(z) (d^n \delta(z)/dz^n)$  where each  $a_n(z)$  is a polynomial in  $z$ . Then, we can continue to represent the operators  $\hat{q}$  and  $\hat{z}$  by (3.6). Let us define an inner product on  $V$  via:

$$\begin{aligned} \langle \Psi(z) | \Phi(z) \rangle &:= \overline{\Psi(\bar{z})} \Phi(z) \circ \mu \\ &= i \int dz \wedge d\bar{z} \mu(z, \bar{z}) \overline{\Psi(\bar{z})} \Phi(z), \end{aligned} \quad (3.20)$$

where  $\mu = \mu(z, \bar{z})$  is a measure to be determined by the reality conditions and where, in the second step, we have used the integral representation (3.19) of the product of holomorphic and anti-holomorphic distributions. Thus, the ansatz is formally the same as the one used earlier. Furthermore, the previous calculations go through step by step because they only use the fact that the states are holomorphic, i.e., are annihilated by the operator  $d/d\bar{z}$ , and the measure  $\mu(z, \bar{z})$  is therefore again given by (3.13). However, now the integral does converge because of the presence of  $\delta$ -distributions in the expressions of our states. Thus, the inner product is well defined for all elements of  $V$ . The full Hilbert space  $\mathcal{H}$  is obtained just by Cauchy completion. Note incidentally that, while we began with the  $\delta$ -distributions with support at  $z = 0$ , the Cauchy completion includes states with support at other points.  $\Psi(z) = \delta(z, z_0)$ , for example, belongs to  $\mathcal{H}$  and represents a coherent state.

As is expected from our motivating remarks, in this representation it is the *annihilation* operator that is represented by  $\hat{z}$ . The vacuum state is simply the normalized state  $\Psi_0(z) = \delta(z)$ . An orthogonal basis in the Hilbert space is provided by the states  $d^n \delta(z)/dz^n$ . [Thus, we could also have let the representation space  $V$  to be the linear span of states of the type  $\sum a_n (d^n \delta(z)/dz^n)$ , where  $a_n$  are constants.] The Hamiltonian is given by

$$\hat{H} = \frac{1}{2} (\hat{z}^* \hat{z} + 1) \equiv -\frac{1}{2} [(z + 2\hbar d/dz)z - 1]. \quad (3.21)$$

Finally, note that, in spite of the appearance of distributions, this representation does diagonalize the operator  $\hat{z}$ ; it acts as the multiplication operator.

To conclude this subsection, we wish to point out that there is in fact an alternate strategy available to quantize the harmonic oscillator with the present reality conditions. Recall that the quantization program requires two new inputs: the choice of the space  $\mathcal{S}$  of elementary classical variables in the

first step and of the representation of the algebra  $\mathcal{A}$  in the fourth step. In this subsection, we used the same elementary variables as in section 3.1 and changed the carrier space  $V$  of the representation of  $\mathcal{A}$  to accommodate the new reality conditions. Alternatively, we could have changed the space  $\mathcal{S}$  itself in step 1. Let us choose  $(q, \bar{z})$  as the elementary variables in place of  $(q, z)$ . Then, one can in fact proceed exactly as in section 3.1, replacing  $z$  everywhere by  $\bar{z}$ . The program can be completed without recourse to distributions – the states are just polynomials in  $\bar{z}$  – and yet the resulting description is equivalent to the one we have obtained here. However, in this representation, it is  $\hat{z} \equiv \hat{z}^*$  that is diagonal rather than  $\hat{z}!$  Thus, although this strategy is viable, it would have led us to the *anti*-self dual representation of  $j = 1$  modes in the Maxwell theory. We are led to consider distributions precisely because we want to retain the self dual representation for the  $j = 1$  modes as well.

### 3.3. SELF DUAL REPRESENTATION

Let us now combine the results of the last two subsections to construct the self dual representation for the quantum Maxwell field. This is the representation in which the self dual connection  ${}^+A_d^T(\mathbf{x})$  – or, equivalently,  $\hat{z}_j(\mathbf{k})$  – is diagonal.

As discussed in the beginning of this section, we are led to use  $z_j(\mathbf{k}), q_j(\mathbf{k})$  as the elementary classical variables in the quantization program of section 2.1. The elementary quantum operators are then  $\hat{z}_j(\mathbf{k}), \hat{q}_j(\mathbf{k})$  and the algebra  $\mathcal{A}$  is generated by their formal sums of formal products subject to the CCR

$$[\hat{q}_i(-\mathbf{k}), \hat{z}_j(\mathbf{k}')] = \delta_{ij} \hbar \delta^3(\mathbf{k}, \mathbf{k}'), \quad (3.22)$$

which mirror the Poisson bracket relations (2.13). The next step is the introduction of the abstract  $\star$ -relations. The reality conditions (2.14) lead us to the relations:

$$\begin{aligned} \hat{q}_j^* &= \hat{q}_j \\ (\hat{z}_1(\mathbf{k}))^* &= -\hat{z}_1(-\mathbf{k}) - 2|\mathbf{k}|\hat{q}_1(-\mathbf{k}), \\ (\hat{z}_2(\mathbf{k}))^* &= -\hat{z}_2(-\mathbf{k}) + 2|\mathbf{k}|\hat{q}_2(-\mathbf{k}). \end{aligned} \quad (3.23)$$

Next, we wish to select a representation of  $\mathcal{A}$ , ignoring for the moment the  $\star$ -relations. Since we want the operators  $\hat{z}_j(\mathbf{k})$  to act by multiplication, we are led to choose for the carrier space  $V$ , the space spanned by holomorphic



distributions  $\Psi(z_j(\mathbf{k}))$  of the type:

$$\begin{aligned} \Psi(z_j(\mathbf{k})) &= h(z_j(\mathbf{k})) + \sum_{n=1}^N \int \frac{d^3k_1}{|\mathbf{k}_1|} \cdots \int \frac{d^3k_n}{|\mathbf{k}_n|} f_{j_1 \dots j_n}(k_1, \dots, k_n) \\ &\times \frac{\delta}{\delta z_{j_1}(\mathbf{k}_1)} \cdots \frac{\delta}{\delta z_{j_n}(\mathbf{k}_n)} \delta(z_1(\mathbf{k})) \delta(z_2(\mathbf{k})), \end{aligned} \quad (3.24)$$

where  $h(z_j(\mathbf{k}))$  is a holomorphic functional of  $z_j(\mathbf{k})$  and where the repeated indices  $j_i$  are summed over  $j_i = 1, 2$ . Thus, at this stage, without pre-judging the issue we allow both holomorphic functions and derivatives of  $\delta$ -distributions for  $j = 1$  as well as  $j = 2$  modes. On this  $V$ , the basic operators  $\hat{q}_j(\mathbf{k})$  and  $\hat{z}_j(\mathbf{k})$  are represented by:

$$\begin{aligned} \hat{q}_i(-\mathbf{k}) \cdot \Psi(z_j(\mathbf{k})) &= \hbar \frac{\delta}{\delta z_i(\mathbf{k})} \Psi(z_j(\mathbf{k})), \\ \hat{z}_i(\mathbf{k}) \cdot \Psi(z_j(\mathbf{k})) &= z_i(\mathbf{k}) \Psi(z_j(\mathbf{k})), \end{aligned} \quad (3.25)$$

so that the CCR (3.22) are satisfied. The second of these equations ensures that we are working in the self dual representation and the first then provides the simplest way to achieve (3.22). Our job now is to select the inner product using the reality conditions. The similarity of the two modes to the two treatments of the harmonic oscillator enables us to follow the procedures of sections 3.1 and 3.2 step by step. The logic of these calculations is straightforward and care is needed only to keep track of which modes are associated with momentum  $\mathbf{k}$  and which are associated with  $-\mathbf{k}$ . Therefore, we shall simply report the results.

The inner product is again expressible as:

$$\begin{aligned} \langle \Psi(z_j) | \Phi(z_j) \rangle &= \int \left[ \prod_j d\mathbb{1}z_j(\mathbf{k}) \wedge d\mathbb{1}\bar{z}_j(\mathbf{k}) \right] \\ &\times \mu(z_j(\mathbf{k}), \bar{z}_j(\mathbf{k})) \overline{\Psi(z_j)} \Phi(z_j), \end{aligned} \quad (3.26)$$

where  $d\mathbb{1}$  is the infinite dimensional exterior derivative on the space spanned by  $(z_j(\mathbf{k}), \bar{z}_j(\mathbf{k}))$ . The reality conditions again lead to a (functional) differential equation. It has a unique solution: Up to an overall constant multiplicative factor, the measure is given by:

$$\begin{aligned} \mu(z_j(\mathbf{k}), \bar{z}_j(\mathbf{k})) &= \exp \left[ - \sum_j \frac{(-1)^j}{4\hbar} \right. \\ &\times \left. \int \frac{d^3\mathbf{k}}{|\mathbf{k}|} (z_j(\mathbf{k}) + \bar{z}_j(-\mathbf{k})) (z_j(-\mathbf{k}) + \bar{z}_j(\mathbf{k})) \right]. \end{aligned} \quad (3.27)$$

A simple calculation shows that, as in section 3.1, the measure “damps correctly” for  $j = 2$  modes so that the normalizable states are just holomorphic functionals  $h(z_2(\mathbf{k}))$  of  $z_2(\mathbf{k})$ . This space is left invariant by the entire algebra  $\mathcal{A}^{(*)}$ . (If we use distributional states as well for these modes, the norm fails to be positive definite.) For the  $j = 1$  modes, on the other hand, as in section 3.2, the only holomorphic functional  $h(z_1(\mathbf{k}))$  that is normalizable is the zero functional. The normalizable states are all distributional in  $z_1(\mathbf{k})$ . Thus, a general normalized state is a superposition of states of the form:

$$\begin{aligned} \Psi(z_j) = & \int \frac{d^3\mathbf{k}_1}{|\mathbf{k}_1|} \cdots \int \frac{d^3\mathbf{k}_n}{|\mathbf{k}_n|} f(\mathbf{k}_1, \dots, \mathbf{k}_n) \\ & \times \frac{\delta}{\delta z_1(\mathbf{k}_1)} \cdots \frac{\delta}{\delta z_1(\mathbf{k}_n)} \delta(z_1(\mathbf{k})) \\ & \times P(z_2(\mathbf{k})) \exp \left[ \frac{1}{4\hbar} \int \frac{d^3\mathbf{k}}{|\mathbf{k}|} (z_2(\mathbf{k})z_2(-\mathbf{k})) \right], \end{aligned} \quad (3.28)$$

where  $P(z_2(\mathbf{k}))$  is a polynomial in  $z_2(\mathbf{k})$ . The norm of this state is given by:

$$\begin{aligned} \|\Psi(z_j)\|^2 = & \left( \int \frac{d^3\mathbf{k}_1}{|\mathbf{k}_1|} \cdots \int \frac{d^3\mathbf{k}_n}{|\mathbf{k}_n|} |f(\mathbf{k}_1, \dots, \mathbf{k}_n)|^2 \right) \\ & \times \left[ \int d\mathbb{1}_{z_2(\mathbf{k})} \wedge d\mathbb{1}_{\bar{z}_2(\mathbf{k})} \right. \\ & \left. \times \exp \left( -\frac{1}{2\hbar} \int \frac{d^3\mathbf{k}}{|\mathbf{k}|} |z_2(\mathbf{k})|^2 \right) |P(z_2(\mathbf{k}))|^2 \right]. \end{aligned} \quad (3.29)$$

Finally, to make contact with the Fock representation, let us write down the explicit expressions of the annihilation operators, the vacuum state and the Hamiltonian. As one might expect from sections 3.1 and 3.2, there is an asymmetry between the two modes. The annihilation operators are now given by  $\hat{z}_1(\mathbf{k})$  for  $j = 1$  and by  $(\hat{z}_2(\mathbf{k}))^*$  for the  $j = 2$  modes. Consequently, there is also an asymmetry in the expression of the vacuum state. The vacuum is given by

$$\Psi_0(z_j) = \delta(z_1(\mathbf{k})) \exp \left( \frac{1}{2\hbar} \int d^3\mathbf{k} |\mathbf{k}|^{-1} z_2(\mathbf{k})z_2(-\mathbf{k}) \right).$$

It is interesting to translate this expression back in terms of the self dual connection  ${}^+A_a^T(\mathbf{x})$  using (2.11). One finds:  $\Psi_0({}^+A^T) = \delta(z_1(\mathbf{k})) \times \exp Y(A_2)$ , where  $Y(A_2)$  is just the U(1) Chern–Simons action of the self dual connection  $A_2$  constructed from  $z_2$ . Thus, on the  $j = 2$  sector, not only is the exponential

of the Chern–Simons action a normalizable state, but it is in fact the vacuum. \* This comes about *only* because we have used the self dual representation. On the  $j = 1$  sector, on the other hand, the exponential of the Chern–Simons functional – being an ordinary holomorphic functional – fails to be normalizable; only the holomorphic *distributions* are normalizable in this sector. Finally, the normal ordered Hamiltonian is:

$$\begin{aligned} \hat{H} &= -\hbar \int d^3\mathbf{k} \left( \hat{z}_1^*(\mathbf{k}) \hat{z}_1(\mathbf{k}) + \hat{z}_2(\mathbf{k}) \hat{z}_2^*(\mathbf{k}) \right) \\ &= -\hbar \int d^3k \left[ - \left( z_1(-\mathbf{k}) + 2\hbar|k| \frac{\delta}{\delta z_1(\mathbf{k})} \right) z_1(\mathbf{k}) \right. \\ &\quad \left. + z_2(\mathbf{k}) \left( -z_2(-\mathbf{k}) + 2\hbar|k| \frac{\delta}{\delta z_2(\mathbf{k})} \right) \right]. \end{aligned} \tag{3.30}$$

Using the argument given in the case of the harmonic oscillator, it is straightforward to establish that, in spite of this apparent asymmetry, this representation is unitarily equivalent to the Bargmann – and hence also the Fock – representation of free photons. The self dual representation is indeed viable within the framework of the general program.

#### 4. Discussion

The analysis of the last two sections raises several interesting issues.

(i) Perhaps the most surprising aspect of this analysis is the appearance of holomorphic distributions. These appear to be indispensable to obtain a representation in which the self dual connection is diagonal. Our treatment of these distributions, however, is rather naive. Presumably there is a systematic mathematical theory that underlies these ideas. It is likely that such a theory would have other applications to quantum theory as well as to other areas of physics.

(ii) It is encouraging that the self dual representation does in fact exist for the Maxwell field because the conceptual ingredients used in its construction are available also in general relativity [6–8]. Indeed, we have followed, step by step, a quantization program which was introduced in the context of quantum general relativity. Furthermore, our basic canonical variables are the direct analogs of the ones used in that program. In general relativity, the corresponding self dual representation has been used to address a number of

In retrospect this is not totally surprising. The Hamiltonian density of the Maxwell field can indeed be written in the form of a product,  ${}^+B^a{}^-B_a$ . On the  $j = 2$  sector,  ${}^-B_a$  acts essentially as an annihilation operator. The state it kills is just the exponential of the Chern–Simons action.

conceptual as well as technical questions. The conceptual problems include the possibility of the gravitational  $CP$  violation and the issue of time [7]. An example of the technical progress is the availability of exact solutions to quantum constraints in Bianchi IX models [13]. The key assumption in these analyses is the existence of a representation which is diagonal in the self dual (gravitational) connection in which the *Hermitian* operator (analogous to)  $\hat{E}_a^T(\mathbf{x})$  can be expressed by a functional derivative. A priori it is not obvious that these assumptions are viable. That they in fact are viable in the well-understood Maxwell theory is therefore reassuring.

(iii) Nowhere in the construction did we carry out a decomposition of fields into their positive and negative frequency parts. The two helicity modes, labelled by  $j$ , arose as a technical by-product in the process of implementing the quantization program, rather than as irreducible representations of the Poincaré group. Indeed, we did not have to appeal anywhere to the Poincaré invariance. Rather, it is the reality conditions that selected for us the inner product and the vacuum. Of course, had we not restricted ourselves to Minkowski space, we might not have been able to carry out the quantization program to completion. In this sense, the existence of the Poincaré invariance has presumably been used indirectly somewhere in the construction. The observation is rather that, since the group does not feature in an explicit way anywhere, one can be hopeful that the program may be successful in other contexts as well. This hope is borne out in 2+1 dimensional quantum general relativity.

(iv) Finally, this example has taught us, rather clearly, an important lesson about the quantization program: the actual imposition of the reality conditions can be *quite* an involved procedure. Even after the elementary classical variables are fixed, one may still have to invent new representations of the algebra of quantum operators to ensure the existence of a sufficiently large physical Hilbert space. There is as yet no systematic procedure available to construct them. In particular, the self dual representation that we were led to for the  $j = 1$  modes seems to fall outside the geometric quantization program since the distributions involved do not appear to arise as polarized cross-sections of a line bundle on the phase space. Is there perhaps a more general framework that we can rely on for guidelines?

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